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The Initial Flow Past an Impulsively Started Sphere at High Reynolds Numbers

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SUMMARY

A solution for the early flow around an impulsively started sphere in a viscous fluid has been developed in powers of the time from the start of the motion. The boundary-layer solution considered by E. Boltze has been extended and solutions of this type have been developed to include the effect of finite Reynolds numbers. For high Reynolds numbers the time series is valid past the time when separation occurs and a number of characteristic flow properties can be calculated with reasonable accuracy.

1. Introduction

The problem of finding the motion set up by a sphere which is started impulsively with uniform velocity in a viscous fluid was first considered by E. Boltze in 1908 [1]. The method used consists of scaling the coordinate normal to the body, the stream function and the vorticity with respect to a parameter which is proportional to the boundary-layer thickness. The stream function and vorticity are then expanded in series in powers of the time after the impulsive start. E. Boltze obtained numerical solutions in powers up to and including t^3 for the boundary-layer equations. The method was subsequently used by Goldstein and Rosenhead in 1936 [2] to obtain solutions for the flow past an impulsively started cylinder. These are analytical solutions of the boundary-layer equations up to and including terms in t^2. Squire in 1954 [3] extended the Goldstein and Rosenhead solutions for cylinders to the axially-symmetric case. The Goldstein and Rosenhead solutions were shown to be in error by Wundt in 1955 [4] and were corrected at that time. In a recent paper Wang [5] has used the method of inner and outer expansions to consider the impulsively started sphere problem for finite Reynolds numbers. This process soon becomes tedious and only functions associated with powers up to t are derived.

The present paper is concerned with the impulsively started sphere problem. Solutions in the form of time series, valid for small times after the impulsive start, are obtained for both the boundary-layer case and also for finite Reynolds numbers. Some analytical solutions are given, but since these rapidly build in complexity, the majority of the solutions were found numerically. The accuracy of the Boltze solutions is improved and the solutions for the boundary-layer case are continued up to t^7.

2. Equations of Motion and Boundary Conditions

A spherical polar coordinate system (r, θ, ϕ) centred at the sphere is chosen. The motion is assumed to be axially symmetric and hence all quantities are independent of the azimuthal coordinate ϕ. Moreover it is assumed that no swirling motion occurs and hence $v_\phi = 0$. The motion may be described by radial and polar components of velocity (v_r, v_θ) in a plane through the axis of symmetry. There is then only one component of vorticity ζ', in the ϕ direction, given by

$$\zeta' = \frac{1}{r}\left\{\frac{\partial}{\partial r}(rv_\theta) - \frac{\partial v_r}{\partial \theta}\right\}. \tag{1}$$

The usual non-dimensional quantities defined by

$$v_\theta = U_\infty V, \quad v_r = U_\infty U, \quad \zeta' = U_\infty \zeta/a, \quad T = at/U_\infty \tag{2}$$

are introduced and the transformation $\xi = \log(r/a)$ is also made, where U_∞ is the velocity of the external stream relative to the sphere, a is the radius of the sphere and T is the time. Equation (1) becomes

$$\frac{\partial V}{\partial \xi} + V - \frac{\partial U}{\partial \theta} = e^{\xi}\zeta. \tag{3}$$

For an incompressible fluid, the equation of continuity is

$$\frac{\partial U}{\partial \xi} + 2U + \frac{\partial V}{\partial \theta} + V\cot\theta = 0. \tag{4}$$

If we eliminate the pressure from the Navier-Stokes equations we obtain the equation for ζ, the scalar vorticity, given by

$$\frac{\partial^2\zeta}{\partial\xi^2} + \frac{\partial\zeta}{\partial\xi} + \cot\theta\frac{\partial\zeta}{\partial\theta} + \frac{\partial^2\zeta}{\partial\theta^2} - \frac{\zeta}{\sin^2\theta} = \frac{R}{2}e^{2\xi}\frac{\partial\zeta}{\partial t} + \frac{R}{2}e^{\xi}\left\{U\frac{\partial\zeta}{\partial\xi} + V\frac{\partial\zeta}{\partial\theta} - U\zeta - V\zeta\cot\theta\right\}, \tag{5}$$

where $R = 2aU_\infty/\nu$ is the Reynolds number, ν being the kinematic viscosity. Equations (3), (4), (5) are the three basic equations that must be solved subject to the conditions

$$U = V = 0, \quad \text{when} \quad \xi = 0, \tag{6}$$

$$U \to \cos\theta, \quad V \to -\sin\theta, \quad \text{as} \quad \xi \to \infty. \tag{7}$$

A further condition used in the subsequent analysis is now derived. This may be termed the integral condition.

The velocity components are written as the series

$$U(\xi, \theta, t) = \sum_{m=1}^{\infty} p_m(\xi, t) P_m(z), \tag{8}$$

$$V(\xi, \theta, t) = \sum_{m=1}^{\infty} q_m(\xi, t) Q_m(z), \tag{9}$$

where $z = \cos\theta$, $P_m(z)$ are the Legendre functions and $Q_m(z) = P_m^{(1)}(z)$ are the associated Legendre functions of order 1. Here we define $P_m^{(1)}(z) = -(1-z^2)^{\frac{1}{2}}(dP_m/dz)$, according to the definition of Abramowitz and Stegun ([6], p. 334). If equations (8) and (9) are substituted into equations (3) and (4), then by the standard methods of orthogonal functions

$$\frac{\partial q_m}{\partial \xi} + q_m - p_m = r_m(\xi, t), \tag{10}$$

$$\frac{\partial p_m}{\partial \xi} + 2p_m - m(m+1)q_m = 0, \tag{11}$$

where

$$r_m(\xi, t) = \frac{2m+1}{2m(m+1)} e^{\xi} \int_0^{\pi} \zeta \sin\theta P_m^{(1)}(\cos\theta)\, d\theta. \tag{12}$$

If q_m is eliminated from equations (11) and (12) then

$$\frac{\partial^2 p_m}{\partial \xi^2} + 3\frac{\partial p_m}{\partial \xi} - (m^2+m-2)p_m = m(m+1)r_m. \tag{13}$$

We now define

$$s_m(\xi, t) = p_m e^{3\xi/2}, \tag{14}$$

and then we obtain

$$\frac{\partial^2 s_m}{\partial \xi^2} - (m+\tfrac{1}{2})^2 s_m = m(m+1)e^{3\xi/2} r_m . \tag{15}$$

From the conditions (7) and the definitions of the various functions it follows that

$$e^{-3\xi/2} s_m \to \delta_m , \quad e^{-3\xi/2} \partial s_m/\partial \xi \to \tfrac{3}{2}\delta_m , \quad \text{as} \quad \xi \to \infty , \tag{16}$$

where

$$\delta_1 = 1 , \quad \delta_m = 0 \qquad (m \neq 1) .$$

The conditions (6) give

$$s_m = \partial s_m/\partial \xi = 0 , \quad \text{when} \quad \xi = 0 , \quad \text{for all} \quad m . \tag{17}$$

If equation (15) is integrated once we obtain

$$\frac{\partial s_m}{\partial \xi} + (m+\tfrac{1}{2}) s_m + C_m e^{(m+\frac{1}{2})\xi} - m(m+1)e^{(m+\frac{1}{2})\xi} \int_0^\xi e^{-(m-1)\xi} r_m d\xi = 0 . \tag{18}$$

The conditions (17) give $C_m = 0$ for all m. The conditions (16) imply that

$$\int_0^\infty e^{-(m-1)\xi} r_m(\xi, t) d\xi = \tfrac{3}{2}\delta_m . \tag{19}$$

This is the integral condition. In the subsequent analysis it replaces the boundary conditions (7), which are not required further. Effectively it imposes a condition on the solution of the vorticity equation (5). The function ζ must also satisfy the condition

$$\zeta \to 0 \quad \text{as} \quad \xi \to \infty . \tag{20}$$

The conditions (6), (19) and (20) are sufficient boundary conditions to solve the problem.

3. Method of Solution

The coordinate ξ normal to the sphere is scaled with respect to

$$k = 2(2t/R)^{\frac{1}{2}} , \tag{21}$$

where k is a parameter proportional to the boundary-layer thickness, according to the equation

$$\xi = kx . \tag{22}$$

The radial velocity is also scaled with respect to k. Likewise, the vorticity is scaled with respect to k to remove the time singularity in it at $t=0$. It is also convenient to make exponential transformations to simplify the analysis. New variables are defined by the equations

$$k\zeta = e^{-2kx} w , \quad U = ke^{-kx} u , \quad V = e^{-kx} v . \tag{23}$$

Equations (5), (4) and (3) then become, respectively

$$\frac{\partial^2 w}{\partial x^2} - 3k \frac{\partial w}{\partial x} + k^2 \left\{ 2w + \cot\theta \frac{\partial w}{\partial \theta} + \frac{\partial^2 w}{\partial \theta^2} - \frac{w}{\sin^2\theta} \right\}$$

$$= 4te^{2kx} \frac{\partial w}{\partial t} - 2e^{2kx} w - 2xe^{2kx} \frac{\partial w}{\partial x} + 4t \left\{ u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial \theta} - wv \cot\theta - 3kuw \right\} , \tag{24}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial \theta} + v \cot\theta + ku = 0 , \tag{25}$$

$$\frac{\partial v}{\partial x} - k^2 \frac{\partial u}{\partial \theta} = w \,. \tag{26}$$

In order to solve these equations the vorticity and velocity components are written as power series in k. Thus we assume

$$w(x, \theta, t) = \sum_{i=0}^{\infty} k^i w_i(x, \theta, t) \,, \tag{27}$$

$$u(x, \theta, t) = \sum_{i=0}^{\infty} k^i u_i(x, \theta, t) \,, \tag{28}$$

$$v(x, \theta, t) = \sum_{i=0}^{\infty} k^i v_i(x, \theta, t) \,. \tag{29}$$

By equating powers of k, an integral condition for each $w_i(x, \theta, t)$ may be deduced from equation (19). All velocity functions must be zero at $x=0$. Each $w_i(x, \theta, t)$, $u_i(x, \theta, t)$ and $v_i(x, \theta, t)$ may be expanded as a series of functions of x, Legendre functions and powers of time according to the expressions

$$w_i(x, \theta, t) = \sum_{m=0}^{\infty} t^m \{g^{(i)}_{m,m+1}(x) Q_{m+1}(z) + g^{(i)}_{m,m-1}(x) Q_{m-1}(z) + \ldots\} \,, \tag{30}$$

$$u_i(x, \theta, t) = \sum_{m=0}^{\infty} t^m \{p^{(i)}_{m,m+1}(x) P_{m+1}(z) + p^{(i)}_{m,m-1}(x) P_{m-1}(z) + \ldots\} \,, \tag{31}$$

$$v_i(x, \theta, t) = \sum_{m=0}^{\infty} t^m \{q^{(i)}_{m,m+1}(x) Q_{m+1}(z) + q^{(i)}_{m,m-1}(x) Q_{m-1}(z) + \ldots\} \,, \tag{32}$$

where the first subscript of the x functions refers to the power of t and the second subscript refers to the Legendre function and is used if there is more than one term for a power of t. Each sum in the braces in equations (30), (31) and (32) terminates when the second subscript becomes negative. For convenience we also define the operators

$$L_m = \frac{d^2}{dx^2} + 2x \frac{d}{dx} + (2-4m) \,, \tag{33}$$

$$L^*_m = \frac{d^2}{dx^2} + 2x \frac{d}{dx} - 4m \,. \tag{34}$$

4. The Boundary-layer Expansion

If the expansions (27), (28), (29) are substituted in equations (24), (25) and (26) and terms of $O(k)$ are neglected, we get the equations

$$\frac{\partial^2 w_0}{\partial x^2} + 2x \frac{\partial w_0}{\partial x} + 2w_0 - 4t \frac{\partial w_0}{\partial t} = 4t \left\{ u_0 \frac{\partial w_0}{\partial x} + v_0 \frac{\partial w_0}{\partial \theta} - w_0 v_0 \cot \theta \right\} , \tag{35}$$

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial \theta} + v_0 \cot \theta = 0 \,, \tag{36}$$

$$\frac{\partial v_0}{\partial x} = w_0 \,. \tag{37}$$

The above equations are the unsteady boundary-layer equations. The expansions (30), (31), (32) are substituted and, by equating coefficients of the powers of t, the equations which result for the boundary-layer functions (written without the zero superscript) are

$$L_m g_{mj} = \Gamma_{mj}(0)\,, \tag{38}$$

$$\frac{dq_{mj}}{dx} = g_{mj}\,, \tag{39}$$

$$\frac{dp_{mj}}{dx} - j(j+1)q_{mj} = 0\,, \tag{40}$$

where Γ_{mj} are defined in the Appendix.

The integral condition for g_0 is

$$\int_0^\infty g_0 dx = \tfrac{3}{2}\,, \tag{41}$$

while for the other vorticity functions we have

$$\int_0^\infty g_{mj} dx = 0\,. \tag{42}$$

The solutions for the g functions must satisfy the conditions (41) and (42) and also, naturally, they must vanish as $x\to\infty$. The solutions for the p and q functions must all satisfy the initial conditions

$$p_{mj}(0) = q_{mj}(0) = 0\,. \tag{43}$$

Exact solutions satisfying these conditions can be obtained for the functions associated with the lower powers of t in the expansions (30), (31) and (32). The expressions soon become complicated, and the functions associated with higher powers of t are obtained numerically.

5. High-order Approximations

By means of higher-order approximations we can calculate the departure from boundary-layer theory for any large, but finite, value of R. In the present method of solution it is relatively easy to calculate higher-order terms and we can describe the procedure generally as follows.

The coefficient of k^n in each of equations (24), (25) and (26) is, respectively,

$$\begin{aligned}\frac{\partial^2 w_n}{\partial x^2} + 2x\frac{\partial w_n}{\partial x} - 2(n-1)w_n - 4t\frac{\partial w_n}{\partial t} &= 3\frac{\partial w_{n-1}}{\partial x} - 2w_{n-2} - \cot\theta\frac{\partial w_{n-2}}{\partial\theta} \\ &- \frac{\partial^2 w_{n-2}}{\partial\theta^2} + \frac{w_{n-2}}{\sin^2\theta} + \sum_{i=1}^{n}\frac{(2x)^i}{i!}\left\{4t\frac{\partial w_{n-i}}{\partial t} + 2(n-i-1)w_{n-i} - 2x\frac{\partial w_{n-i}}{\partial x}\right\} \\ &+ 4t\sum_{i=0}^{n}\left\{u_i\frac{\partial w_{n-i}}{\partial x} + v_i\frac{\partial w_{n-i}}{\partial\theta} - v_i w_{n-i}\cot\theta\right\} - 12t\sum_{i=0}^{n-1}u_i w_{n-i-1}\,,\end{aligned} \tag{44}$$

$$\frac{\partial u_n}{\partial x} + \frac{\partial v_n}{\partial\theta} + v_n\cot\theta + u_{n-1} = 0\,, \tag{45}$$

$$\frac{\partial v_n}{\partial x} = w_n + \frac{\partial u_{n-2}}{\partial\theta}\,. \tag{46}$$

In these equations a negative subscript indicates that the function is zero. Substitution of the expansions (30), (31) and (32) and equating coefficients of powers of t gives differential equations for the k^n functions. These can be written

$$D_n^* g_{mj}^{(n)} = \Gamma_{mj}(n) + \beta_{mj}(n) + \frac{3dg_{mj}^{(n-1)}}{dx} + \{j(j+1)-2\} g_{mj}^{(n-2)}$$

$$+ \sum_{i=1}^{n} \frac{(2x)^i}{i!} \left\{ 4mg_{mj}^{(n-i)} + 2(n-i-1) g_{mj}^{(n-i)} - 2x \frac{dg_{mj}^{(n-i)}}{dx} \right\}, \quad (47)$$

$$\frac{dq_{mj}^{(n)}}{dx} = g_{mj}^{(n)} + p_{mj}^{(n-2)}, \quad (48)$$

$$\frac{dp_{mj}^{(n)}}{dx} - j(j+1) q_{mj}^{(n)} + p_{mj}^{(n-1)} = 0, \quad (49)$$

where a negative superscript indicates that the function is zero and the operator D_n^* is defined by

$$\begin{aligned} D_n^* &= L_{m+(n-1)/2}^* \qquad n \text{ odd}, \\ &= L_{m+n/2} \qquad n \text{ even}. \end{aligned} \quad (50)$$

The $\Gamma_{ij}(n)$ and $\beta_{ij}(n)$ are defined in the Appendix.

The integral condition for the k^n vorticity functions is, in general,

$$\int_0^\infty g_{mj}^{(n)} dx = -\sum_{i=1}^{n} \int_0^\infty \frac{(-jx)^i}{i!} g_{mj}^{(n-i)} dx. \quad (51)$$

The vorticity functions must all vanish as $x \to \infty$ and the p and q functions must vanish when $x=0$. As in the case of the boundary-layer expansion, exact solutions can be obtained for the functions associated with the lower powers of t and numerical methods must be used to obtain the functional coefficients of the higher powers of t because the functions become too complicated. To complete the theory we shall now briefly describe the numerical procedure.

6. Numerical Methods

The differential equations for the vorticity functions $g_{mj}^{(i)}$ are of the form

$$g'' + 2xg' - \alpha g = K(x) \quad (52)$$

If we define

$$g(x) = e^{-x^2/2} G(x), \quad (53)$$

then G satisfies the equation

$$G'' - r(x) G = f(x), \quad (54)$$

where

$$r(x) = x^2 + 1 - \alpha, \quad f(x) = e^{x^2/2} K(x). \quad (55)$$

The equation (54) may be written in finite-difference form at a grid point $x = ih$ as

$$(1-h^2 r_{i-1}/12) G_{i-1} + (1-h^2 r_{i+1}/12) G_{i+1} - (2+5h^2 r_i/6) G_i$$

$$- \frac{h^2}{12}(f_{i-1} + 10f_i + f_{i+1}) + c' G_i = 0, \quad (56)$$

where

$$c' = \delta^6/240 - 13\delta^8/15120 + \ldots$$

and h is the step size used. Neglect of the correction term then gives an accurate approximation to the differential equation (see Fox [7], p. 68).

All vorticity functions must vanish at infinity. In practice, a finite field length must be chosen and this boundary condition imposed at this finite distance. Since there is no intrinsic way to choose the finite length, the field was increased until no significant change was observed in the solution. A length of $x=6$ was finally used with a step size of $h=0.05$. The step size was halved as a check on the accuracy and no appreciable change resulted in the solutions.

A boundary condition for the vorticity is required at $x=0$. This is obtained by making each vorticity function satisfy the appropriate integral condition. The boundary condition at $x=0$ is arbitrarily put equal to 1 and a solution of equation (52) is obtained. If y_N denotes this solution, the actual solution y_T must be y_N plus a multiple of the homogeneous solution y_H, thus

$$y_T = y_N + C y_H \,. \tag{57}$$

The homogeneous solutions of $L_m=0$ which vanish as $x\to\infty$ are

$$H_0^{(m)}(x) = \mathrm{e}^{-x^2/2} D_{-2m}(\sqrt{2}x)\,, \tag{58}$$

where

$$\int_0^\infty \mathrm{e}^{-x^2/2} D_{-2m}(\sqrt{2}x)dx = \frac{\sqrt{\pi}}{2^{m+1}m!}\,.$$

The similar homogeneous solutions of $L_m^*=0$ are

$$H_0^{(m)*}(x) = \mathrm{e}^{-x^2/2} D_{-2m-1}(\sqrt{2}x)\,, \tag{59}$$

where

$$\int_0^\infty \mathrm{e}^{-x^2/2} D_{-2m-1}(\sqrt{2}x)dx = \frac{1}{2^{m+1}\Gamma(\frac{3}{2}+m)}\sqrt{\frac{\pi}{2}}\,.$$

Here $D_\alpha(x)$ are the parabolic cylinder functions.

The constant C in equation (57) may now be determined since y_T must satisfy the appropriate integral condition obtained from (19) by equating powers of t and k. To obtain y_T from (57), y_H would have to be evaluated numerically, since the parabolic cylinder functions are not tabulated well enough for use. However it is just as easy to determine the correct boundary condition at $x=0$ and solve the problem again. Since

$$H_0^{(m)}(0) = 2^{m-1}\frac{\Gamma(m)}{\Gamma(2m)}\,, \qquad H_0^{(m)*}(0) = \frac{\sqrt{\pi}}{2^{m+\frac{1}{2}}\Gamma(m+1)}$$

the correct boundary condition $y_T(0)$ may be determined once the value of C has been calculated.

When the finite-difference approximation (56) to equation (54) is used at each internal point, a matrix problem

$$AG = B$$

results, where A is a tridiagonal matrix. This is solved by the direct method given by Rosser [8]. Finally, once the vorticity has been determined for a particular $g_{mj}^{(n)}$ the corresponding polar velocity $q_{mj}^{(n)}$ and radial velocity $p_{mj}^{(n)}$ may be determined by a step-by-step integration from $x=0$, where both must be zero. The ordinary Simpson formula was used in general, but to start the integration the formula

$$\int_{x_0}^{x_1} y\,dx = \frac{h}{24}(9y_0 + 19y_1 - 5y_2 + y_3)$$

was used. The sixth-order differentiation formulae given by Bickley [9] were used to evaluate the vorticity derivatives.

7. Exact Solutions

The boundary-layer vorticity functions have been derived by exact analysis up to and including the t^2 functions. As shown by Squire [3], these solutions may be derived from the Goldstein-

Rosenhead solutions [2], but the following were derived from the equations (38), (39) and (40) subject to the conditions (41), (42) and (43). The vorticity functions are listed, along with some of the associated velocity functions, below. The functions associated with the zero power of t are

$$g_0(x)=\frac{3}{\sqrt{\pi}}e^{-x^2},\quad q_0(x)=\tfrac{3}{2}\operatorname{erf}x,\quad p_0(x)=3\left\{x\operatorname{erf}x+\frac{e^{-x^2}}{\sqrt{\pi}}-\frac{1}{\sqrt{\pi}}\right\}. \tag{61}$$

The vorticity function associated with the first power of t is

$$g_1(x)=\frac{3}{\sqrt{\pi}}x^2e^{-x^2}\operatorname{erf}x+\frac{3}{2\sqrt{\pi}}e^{-x^2}\operatorname{erf}x+\frac{3}{\pi}xe^{-2x^2}-\frac{4}{\pi}xe^{-x^2}$$
$$+A\left\{x\operatorname{erf}x-x+\frac{e^{-x^2}}{\sqrt{\pi}}\right\}, \tag{62}$$

where $A=2/\pi-3$.

These functions and the velocity functions associated with the first power of t may be derived from Squire [3], as may the functions associated with the power of t^2. They also have been derived from basic principles by Walker [10]. The functions given here were the only ones required to derive exact solutions for the higher-order terms.

The functions of order k have been derived up to and including the functions associated with the term kt. The terms of order k associated with the zero power of t are

$$g_0^{(1)}=\tfrac{3}{2}(1-\operatorname{erf}x)-\frac{3x^2}{\sqrt{\pi}}e^{-x^2}+\frac{3xe^{-x^2}}{\sqrt{\pi}}, \tag{63}$$

$$q_0^{(1)}=-\tfrac{3}{2}\left\{x\operatorname{erf}x-x-\frac{x^2e^{-x^2}}{\sqrt{\pi}}+\frac{e^{-x^2}}{\sqrt{\pi}}-\frac{1}{\sqrt{\pi}}\right\}, \tag{64}$$

$$p_0^{(1)}=-3x^2\operatorname{erf}x-\tfrac{3}{4}\operatorname{erf}x-\frac{9}{2\sqrt{\pi}}xe^{-x^2}+\frac{3x^2}{2}+\frac{6x}{\sqrt{\pi}}. \tag{65}$$

The vorticity function associated with the term kt is

$$g_1^{(1)}=B'(2x^2+1)(\operatorname{erf}x-1)+\frac{2B'xe^{-x^2}}{\sqrt{\pi}}+A\left\{\tfrac{3}{4}-\tfrac{3}{4}\operatorname{erf}x-\frac{xe^{-x^2}}{2\sqrt{\pi}}\right\}$$
$$-\tfrac{3}{4}(2x^2+1)\{(\operatorname{erf}x)^2-1\}-\frac{1}{2\sqrt{\pi}}\{6x^5+x^3+3x\}e^{-x^2}\operatorname{erf}x$$
$$-\frac{1}{\pi}(3x^4-x^2-1)e^{-2x^2}+\frac{1}{5\pi}\{20x^4+6x^2-18\}e^{-x^2}+\frac{1}{4\sqrt{\pi}}\{4x^3+3x\}e^{-x^2}, \tag{66}$$

where

$$B'=-6\sqrt{2}+\frac{57}{16}+\frac{1}{2\pi}.$$

The first functions of order k^2 are

$$g_0^{(2)}=\frac{3}{2\sqrt{\pi}}x^6e^{-x^2}-\frac{19}{4\sqrt{\pi}}x^4e^{-x^2}, \tag{67}$$

$$q_0^{(2)}=\frac{3x^2}{2}\operatorname{erf}x+\tfrac{3}{8}\operatorname{erf}x-\frac{3}{4\sqrt{\pi}}x^5e^{-x^2}+\frac{1}{2\sqrt{\pi}}x^3e^{-x^2}+\frac{9}{4\sqrt{\pi}}xe^{-x^2}-\frac{3x}{\sqrt{\pi}}, \tag{68}$$

$$p_0^{(2)}=2x^3\operatorname{erf}x+\tfrac{3}{2}x\operatorname{erf}x+\frac{3}{4\sqrt{\pi}}x^4e^{-x^2}+\frac{3}{\sqrt{\pi}}x^2e^{-x^2}-\frac{x^3}{2}-\frac{6x^2}{\sqrt{\pi}}. \tag{69}$$

Finally, the first vorticity function of order k^3 is

$$g_0^{(3)} = -\frac{1}{4\sqrt{\pi}}(2x^9 - 13x^7 + 13x^5)e^{-x^2} . \tag{70}$$

8. Results

The boundary-layer functions were evaluated numerically up to t^7, the k functions to kt^6, the k^2 functions to k^2t^4, the k^3 functions to k^3t^3, the k^4 functions to k^4t^2 and the k^5 functions to k^5t. The values of the vorticity functions on the surface of the sphere ($x=0$) are given in Table 1,

TABLE 1

	Boundary Layer $i=0$	k $i=1$	k^2 $i=2$	k^3 $i=3$	k^4 $i=4$	k^5 $i=5$
$g_0^{(i)}$	1.6926	1.5000	0.	0.	0.	0.
$g_1^{(i)}$	−1.3334	2.9135	− 7.624	20.85	− 58.35	166.8
$g_{21}^{(i)}$	0.0778	0.7251	− 4.480	18.38	− 65.83	
$g_{23}^{(i)}$	−0.1507	1.8249	−12.58	68.45	−331.8	
$g_{32}^{(i)}$	0.0406	1.1319	−14.30	107.7		
$g_{34}^{(i)}$	0.0161	0.3533	− 7.638	83.47		
$g_{41}^{(i)}$	−0.0476	0.5757	− 6.786			
$g_{43}^{(i)}$	−0.0039	0.5523	−12.78			
$g_{45}^{(i)}$	0.0221	−0.4118	1.652			
$g_{52}^{(i)}$	−0.0140	0.2790				
$g_{54}^{(i)}$	−0.0130	−0.1063				
$g_{56}^{(i)}$	0.0056	−0.3683				
$g_{61}^{(i)}$	0.0092	−0.0065				
$g_{63}^{(i)}$	0.0087	−0.1572				
$g_{65}^{(i)}$	−0.0049	−0.2944				
$g_{67}^{(i)}$	−0.0020	0.6496				
$g_{72}^{(i)}$	0.0001					
$g_{74}^{(i)}$	0.0080					
$g_{76}^{(i)}$	0.0016					
$g_{78}^{(i)}$	−0.0020					

while the values of the vorticity derivatives at $x=0$ are given in Table 2. Most of the properties of physical interest can be worked out from these.

The drag on the sphere is

$$D = -\pi a^2 \int_0^\pi p_0 \sin 2\theta \, d\theta - 2\pi\rho\nu U_\infty a \int_0^\pi \zeta_0 \sin^2\theta \, d\theta , \tag{71}$$

where p_0, ζ_0 are the pressure and vorticity evaluated at $x=0$. The drag coefficient $C_D = D/(\pi\rho U_\infty^2 a^2)$ is composed of two parts, C_p and C_f. From (71), the friction drag coefficient is

$$C_f = \frac{4}{R}\int_0^\pi \sin\theta \, \zeta_0 P_1^{(1)}(\cos\theta) d\theta .$$

If ζ_0 is written as a series of Legendre functions

$$\zeta_0 = \frac{1}{k}\sum_{m=1}^{\infty} G_m(0, t) Q_m(z) ,$$

TABLE 2

	Boundary Layer $i=0$	k $i=1$	k^2 $i=2$	k^3 $i=3$	k^4 $i=4$	k^5 $i=5$
$dg_0^{(i)}/dx$	0.	0.	0.	0.	0.	0.
$dg_1^{(i)}/dx$	3.0000	−7.6604	24.36	− 73.81	223.2	−681.5
$dg_{21}^{(i)}/dx$	0.	−2.8834	17.53	− 75.66	286.1	
$dg_{23}^{(i)}/dx$	0.	−4.8615	42.76	−261.1	1372.	
$dg_{32}^{(i)}/dx$	0.	−4.6965	59.40	−472.1		
$dg_{34}^{(i)}/dx$	0.	−1.4299	29.83	−349.6		
$dg_{41}^{(i)}/dx$	0.	−0.9838	22.89			
$dg_{43}^{(i)}/dx$	0.	−2.0028	54.94			
$dg_{45}^{(i)}/dx$	0.	1.0256	−4.41			
$dg_{52}^{(i)}/dx$	0.	−0.5116				
$dg_{54}^{(i)}/dx$	0.	0.9795				
$dg_{56}^{(i)}/dx$	0.	1.2690				
$dg_{61}^{(i)}/dx$	0.	−0.1736				
$dg_{63}^{(i)}/dx$	0.	0.4770				
$dg_{65}^{(i)}/dx$	0.	1.6112				
$dg_{67}^{(i)}/dx$	0.	−3.0453				
$dg_{72}^{(i)}/dx$	0.					
$dg_{74}^{(i)}/dx$	0.					
$dg_{76}^{(i)}/dx$	0.					
$dg_{78}^{(i)}/dx$	0.					

then

$$C_f = \frac{16}{3Rk} G_1(0, t). \tag{72}$$

The pressure drag coefficient is

$$C_p = -\frac{1}{\rho U_\infty^2} \int_0^\pi p_0 \sin 2\theta \, d\theta .$$

From the equations of motion it is found that

$$\left(\frac{\partial p}{\partial \theta}\right)_{\xi=0} = \frac{\rho \nu U_\infty}{a} \left(\zeta + \frac{\partial \zeta}{\partial \xi}\right)_{\xi=0},$$

and if we substitute for ζ and integrate with respect to θ, the result

$$p_0 = \frac{\rho \nu U_\infty}{ka} \sum_{m=1}^{\infty} \left(G_m + \frac{1}{k}\frac{\partial G_m}{\partial x}\right)_{x=0} P_m(z),$$

is obtained, and hence

$$C_p = -\frac{8}{3Rk} \left(G_1(0, t) + \frac{1}{k}\frac{\partial G_1}{\partial x}(0, t)\right). \tag{73}$$

The quantity C_f may now be written as a time series $E(0, t)$ where

$$E(x, t) = \frac{16}{3Rk} \sum_{i=0}^{\infty} k^i \{g_0^{(i)} + t^2 g_{21}^{(i)} + t^4 g_{41}^{(i)} + t^6 g_{61}^{(i)} + \ldots\},\tag{74}$$

and C_p may be derived from equation (74) as

$$C_p = \tfrac{1}{2}\left(E - \frac{1}{k}\frac{\partial E}{\partial x}\right)_{x=0}, \tag{75}$$

where the functions $g_{mj}^{(i)}$ and their derivatives at $x=0$ are given in Tables 1 and 2. The temporal development of the drag is given in Table 3 for various Reynolds numbers.

Separation first occurs at $\theta=0$ when $(\partial w/\partial\theta)_{\theta=0}$ becomes zero. This gives the equation

$$\sum_{i=0}^{\infty} k^i \{g_0^{(i)} + 3t_s g_1^{(i)} + t_s^2(g_{21}^{(i)} + 6g_{23}^{(i)}) + t_s^3(3g_{32}^{(i)} + 10g_{34}^{(i)}) + t_s^4(g_{41}^{(i)} + 6g_{43}^{(i)} + 15g_{45}^{(i)}) + t_s^5 \ldots\}_{x=0} = 0. \tag{76}$$

Equation (76) is solved for t_s, the time of separation, by Newton's method. Some results for

TABLE 3

R		$t=.05$	$t=.1$	$t=.2$	$t=.4$	$t=.6$
40	C_f	2.46				
	C_p	1.23				
	C_D	3.69				
200	C_f	1.05	.75	.55	.40	
	C_p	.53	.38	.28	.22	
	C_D	1.58	1.14	.83	.62	
1000	C_f	.46	.33	.23	.17	.15
	C_p	.23	.17	.12	.10	.096
	C_D	.69	.49	.36	.27	.25

TABLE 4

R	t_s
400	.584
500	.554
700	.523
1000	.499
10^4	.426
10^5	.406
∞	.396

various Reynolds numbers are given in Table 4. Equation (76) fails to have a positive root for Reynolds numbers below approximately 350 and this is because the separation times for these Reynolds numbers are greater than $t=0.6$. The series is not thought to be meaningful at times higher than 0.6 and certainly not for these Reynolds numbers at high times. Boltze gives t_s for $R=\infty$ as 0.392. His series was computed up to t^3 while the present series is computed to t^7.

The Blasius series [11] for axially-symmetric bodies gives the separation angle for steady flow as 70.4°. In Fig. 1 the separation angle is plotted as a function of time for several Reynolds numbers. The result for $R=\infty$ is probably not valid much beyond $t=1$. As a further comparison, the Blasius series gives

$$-\frac{1}{\rho U_\infty^2}\left(\frac{\partial\tau_0}{\partial\theta}\right)_{\theta=\pi} = \frac{3.4085}{R^{\frac{1}{2}}},$$

where τ_0 is the local skin friction. Evaluation of this quantity from the boundary-layer series indicates that it gradually decreases to $3.395R^{-\frac{1}{2}}$ at $t=1$ and passes through the Blasius value at $t=0.875$.

It is difficult to assess the upper limit in time for which the time series will converge. In the boundary-layer case, after $t=0.6$ the equi-vorticity lines at some distance from the body develop physically unreal indentations in the wake region. For this reason, it is felt that the upper limit for convergence within the whole field is $t=0.6$. However, close to the body the series is thought to be meaningful for higher times as evidenced by the comparisons with the Blasius series for steady flow.

For finite Reynolds numbers, the series should also converge in powers of k for the results to be valid. For a particular Reynolds number, an estimate of the region of convergence may be obtained by comparing successive approximations to the drag coefficients in powers of k at various times. Thus $R=1000$ can be considered valid at least until $t=0.6$ while $R=40$ can only be considered valid as far as $t=0.05$.

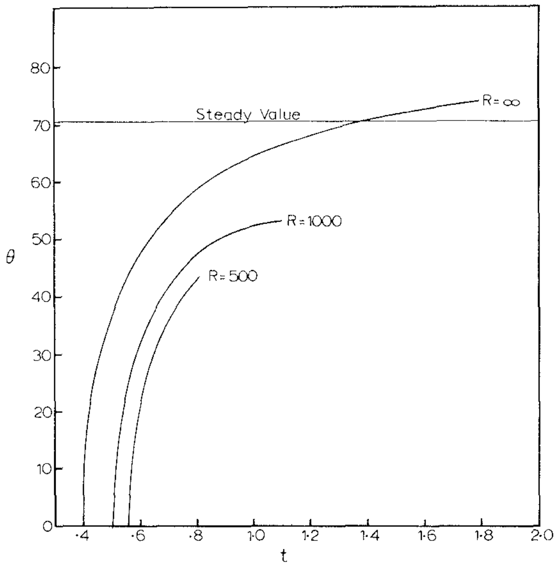

Figure 1. Separation angle versus time.

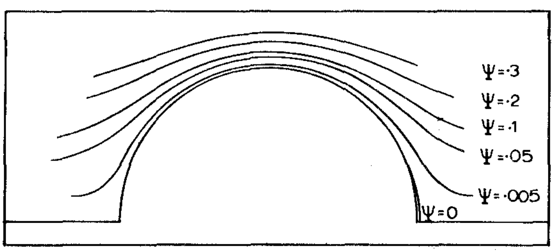

Figure 2. Streamlines for $R=1000$ at $t=0.6$.

To plot the streamlines the dimensionless axially-symmetric stream function ψ is introduced by the equations

$$U = \frac{1}{e^{2\xi}\sin\theta}\frac{\partial\psi}{\partial\theta}, \qquad V = -\frac{1}{e^{2\xi}\sin\theta}\frac{\partial\psi}{\partial\xi}. \tag{77}$$

It may be deduced from either of these equations that

$$\psi = e^{2\xi}Q_1(z)\sum_{m=1}^{\infty}\frac{p_m(\xi,t)}{m(m+1)}Q_m(z). \tag{78}$$

The flow pattern for Reynolds number 1000 at $t=0.6$ is plotted in Fig. 2. The streamlines in the boundary-layer case are also plotted in Figs. 3(a) to 3(c). In order to show the development

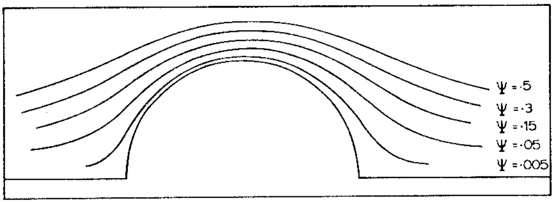

Figure 3(a). Streamlines for boundary-layer flow at $t=0.4$.

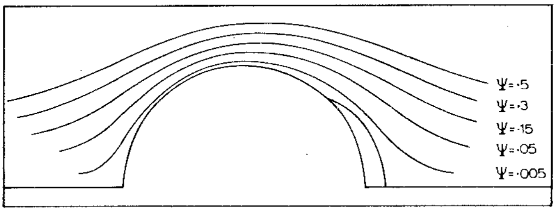

Figure 3(b). Streamlines for boundary-layer flow at $t=0.6$.

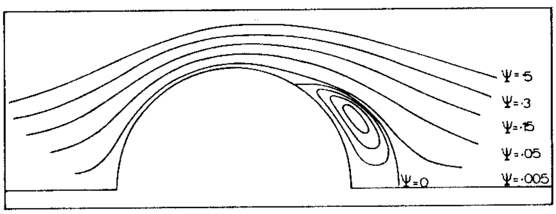

Figure 3(c). Streamlines for boundary-layer flow at $t=0.8$.
(Enclosed streamlines, starting from the centre, are $\psi=-0.005$, $\psi=-0.003$, $\psi=-0.001$).

with time clearly, R has been put equal to 100 in this case. The temporal development of surface vorticity for $R=1000$ is shown in Fig. 4 and for $R=\infty$ in Fig. 5.

The pressure distribution on the surface is given by

$$\frac{p-p_\pi}{\frac{1}{2}\rho U_\infty^2}=\frac{4}{R}\int_\pi^\theta\left(\zeta+\frac{\partial\zeta}{\partial\xi}\right)_{\xi=0}d\theta\,,$$

where p_π is the pressure at $\theta=\pi$. It follows that

$$\frac{p-p_\pi}{\frac{1}{2}\rho U_\infty^2}=-\frac{4}{kR}\sum_{m=1}^{\infty}\left\{G_m+\frac{1}{k}\frac{\partial G_m}{\partial x}\right\}_{x=0}\{(-1)^m-P_m(z)\}\,. \tag{79}$$

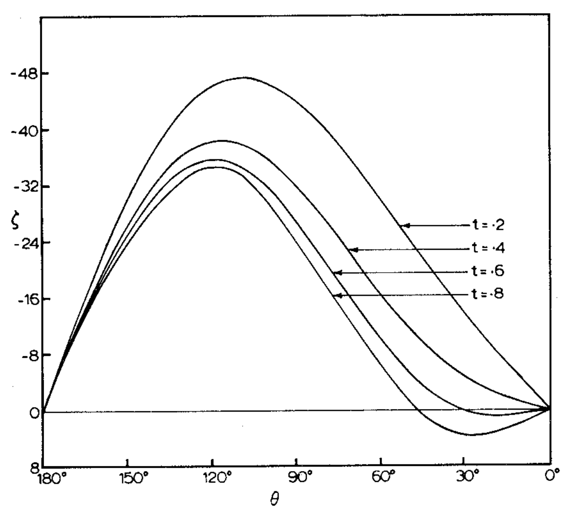

Figure 4. Temporal development of the vorticity on the surface of the sphere for $R=1000$.

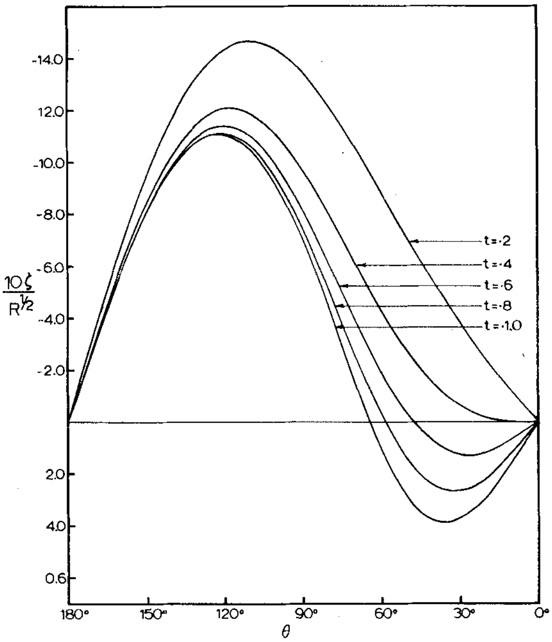

Figure 5. Temporal development of the vorticity on the surface of the sphere for boundary-layer flow.

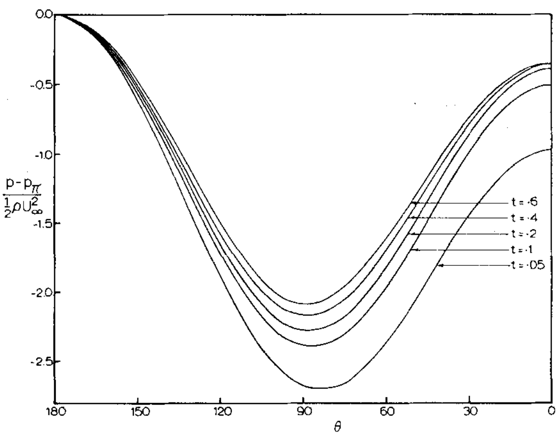

Figure 6. Temporal development of the pressure on the surface of the sphere for $R = 1000$.

The temporal development of surface pressure is shown in Fig. 6 for $R = 1000$. Equation (79) indicates a $t^{-\frac{1}{2}}$ singularity in the surface pressure at $t = 0$, but if t is small and non-zero then the right side tends to $9(\cos 2\theta - 1)/8$ as $R \to \infty$, which is the pressure distribution for potential flow.

In summary, the series method gives solutions for all Reynolds numbers which are valid at early times after the impulsive start. Although the region of convergence is quite small for low Reynolds number for high Reynolds numbers it is valid for times past the occurrence of separation. This series may be used to check the results of a time-dependent integration at the early stages or it could be used to start the integration off. It gives the correct solution at small times and moreover eliminates part of the difficulty of having to take a large number of time steps near $t = 0$ which was encountered by Rimon and Cheng [12] in their numerical study of the problem.

Acknowledgement

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Appendix

Due to the copious quantities of terms that appear on the right hand side of the differential equations, it is necessary to devise a mechanism to write the equations in a more compact form. Since similar groups of terms appear repeatedly this is done by defining the functions $\Gamma_{ij}(n)$ and $\beta_{ij}(n)$,

$$\Gamma_0(n) = \beta_0(n) = 0 , \tag{A-1}$$

$$\Gamma_1(n) = \tfrac{4}{3} \sum_{j=0}^{n} p_0^{(j)} \frac{dg_0^{(k)}}{dx} , \tag{A-2}$$

$$\beta_1(n) = -4 \sum_{j=1}^{n-1} p_0^{(j)} g_0^{(m)} , \tag{A-3}$$

$$\Gamma_{21}(n) = \tfrac{4}{5} \sum_{j=0}^{n} \left\{ 3\, p_0^{(j)} \frac{dg_1^{(k)}}{dx} - p_1^{(j)} \frac{dg_0^{(k)}}{dx} + 12\, q_0^{(j)} g_1^{(k)} \right\} , \tag{A-4}$$

$$\beta_{21}(n) = -\tfrac{4}{5} \sum_{j=0}^{m-1} \{ 9\, p_0^{(j)} g_1^{(m)} - 3\, p_1^{(j)} g_0^{(m)} \} , \tag{A-5}$$

$$\Gamma_{23}(n) = \tfrac{4}{5} \sum_{j=0}^{n} \left\{ 2\, p_0^{(j)} \frac{dg_1^{(k)}}{dx} + p_1^{(j)} \frac{dg_0^{(k)}}{dx} - 2\, q_0^{(j)} g_1^{(k)} \right\} , \tag{A-6}$$

$$\beta_{23}(n) = -\tfrac{4}{5} \sum_{j=0}^{n-1} \{ 6\, p_0^{(j)} g_1^{(m)} + 3\, p_1^{(j)} g_0^{(m)} \} , \tag{A-7}$$

$$\begin{aligned} \Gamma_{32}(n) = \tfrac{4}{21} \sum_{j=0}^{n} \Bigg\{ & 7\, p_0^{(j)} \frac{dg_{21}^{(k)}}{dx} + 12\, p_0^{(j)} \frac{dg_{23}^{(k)}}{dx} + 3\, p_1^{(j)} \frac{dg_1^{(k)}}{dx} + 7\, p_{21}^{(j)} \frac{dg_0^{(k)}}{dx} + \\ & - 3\, p_{23}^{(j)} \frac{dg_0^{(k)}}{dx} + 60\, q_0^{(j)} g_{23}^{(k)} + 36\, q_1^{(j)} g_1^{(k)} \Bigg\} , \end{aligned} \tag{A-8}$$

$$\beta_{32}(n) = -\tfrac{4}{21} \sum_{j=0}^{n-1} \{ 21\, p_0^{(j)} g_{21}^{(m)} + 36\, p_0^{(j)} g_{23}^{(m)} + 9\, p_1^{(j)} g_1^{(m)} + 21\, p_{21}^{(j)} g_0^{(m)} - 9\, p_{23}^{(j)} g_0^{(m)} \} , \tag{A-9}$$

$$\Gamma_{34}(n) = \tfrac{4}{35} \sum^{n} \left\{ 15\, p_0^{(j)} \frac{dg_{23}^{(k)}}{dx} + 9\, p_1^{(j)} \frac{dg_1^{(k)}}{dx} + 5\, p_{23}^{(j)} \frac{dg_0^{(k)}}{dx} - 30\, q_0^{(j)} g_{23}^{(k)} - 18\, q_1^{(j)} g_1^{(k)} \right\} , \tag{A-10}$$

$$\beta_{34}(n) = -\tfrac{4}{35} \sum_{j=0}^{n-1} \{ 45\, p_0^{(j)} g_{23}^{(m)} + 27\, p_1^{(j)} g_1^{(m)} + 15\, p_{23}^{(j)} g_0^{(m)} \} , \tag{A-11}$$

where $k = n - j$ in $\Gamma_{ij}(n)$ and $m = n - j - 1$ in $\beta_{ij}(n)$.

The terms that appear on the right side of equations (38) and (47) for functions associated with powers of t greater than t^3 are not given here, due to space limitations. They are given by Walker [10].

REFERENCES

[1] E. Boltze, Grenzschichten an Rotationskörpern in Flüssigkeiten mit kleiner Reibung, *Thesis*, Göttingen (1908).
[2] S. Goldstein and L. Rosenhead, Boundary Layer Growth, *Proc. Camb. Phil. Soc.*, 32 (1936) 392–401.
[3] L. C. Squire, Boundary Layer Growth in Three Dimensions, *Phil. Mag.* (7), 45 (1954) 1272–83.
[4] H. Wundt, Wachstum der laminaren Grenzschicht an schräg angeströmten Zylindern bei Anfahrt aus der Ruhe, *Ing.-Arch.*, 23 (1955) 212.
[5] C.-Y. Wang, The Impulsive Starting of a Sphere, *Quart. Appl. Math.*, 27 (1969) 273–277.
[6] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York (1968).
[7] L. Fox, *Numerical solution of Two-Point Boundary Problems in Ordinary Differential Equations*, Oxford (1957).
[8] J. B. Rosser, The Direct Solution of Difference Analogs of Poisson's equation, *TSR No. 797* Mathematics Research Centre, Madison, Wisconsin (1967).
[9] W. G. Bickley, Formulae for Numerical Differentiation, *Math. Gaz.*, 25 (1941) 19–27.
[10] J. D. A. Walker, The Flow of a Viscous Fluid Past a Sphere, *Ph. D. Thesis*, University of Western Onatrio, London, Ontario (1971).
[11] H. Schlichting, *Boundary Layer Theory*, Pergamon Press (1955).
[12] Y. Rimon and S. I. Cheng, Numerical Solution of a Uniform Flow over a Sphere at Intermediate Reynolds Numbers, *Physics of Fluids*, 12 (1969) 949–959.

Journal of Engineering Math., Vol. 5 (1971) 263–278